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ECE574 – Discrete-Time Signal Processing
Reading Summary, Chapter 2

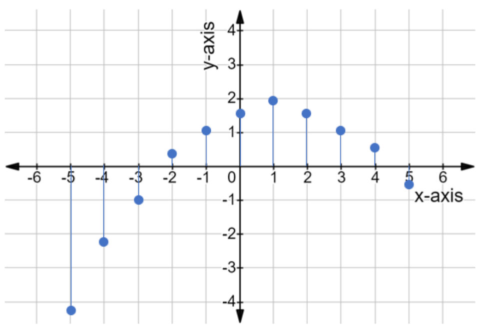
2.0 Introduction

Discrete-Time vs. Digital:

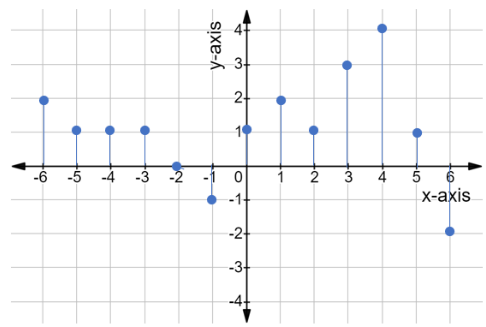
Discrete-Time	**Digital**
The independent variable (most commonly time) is represented by a sequence of numbers of a fixed interval.	Both the independent variable and dependent variable are represented by a sequence of numbers of a fixed interval.

Discrete-Time and Digital Signal examples are shown below:

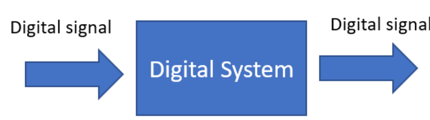
Discrete-Time:
Only limited to vertical lines



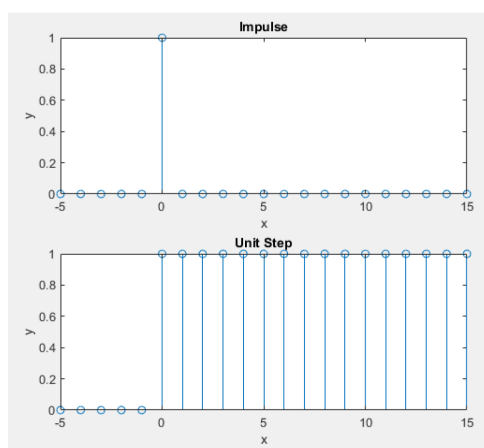
Digital:
Limited to vertical lines and horizontal lines



Discrete-Time Systems and Digital Systems are defined by their inputs and outputs being both either Discrete-Time Signals or Digital Signals.



2.1 Discrete-Time Signals



Discrete-Time Signal x[x] is sequence for all integers n.

No result for non-integer n, undefined.

Unit Sample Sequence $\boldsymbol{\delta}$[n]: 1 at n=0, 0 otherwise.
Unit Step u[n] = 1 at n>=0, 0 otherwise.

$$\text{Or, } u[n] = \sum_{k=-0}^{\infty} \delta[n-k]$$

Any sequence: x[n] = a1* $\boldsymbol{\delta}$[n-1] + a2* $\boldsymbol{\delta}$[n-2]+...
where a1, a2 are magnitude at integer n.

$$\text{or,} \quad x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \boldsymbol{\delta}[n-k]$$

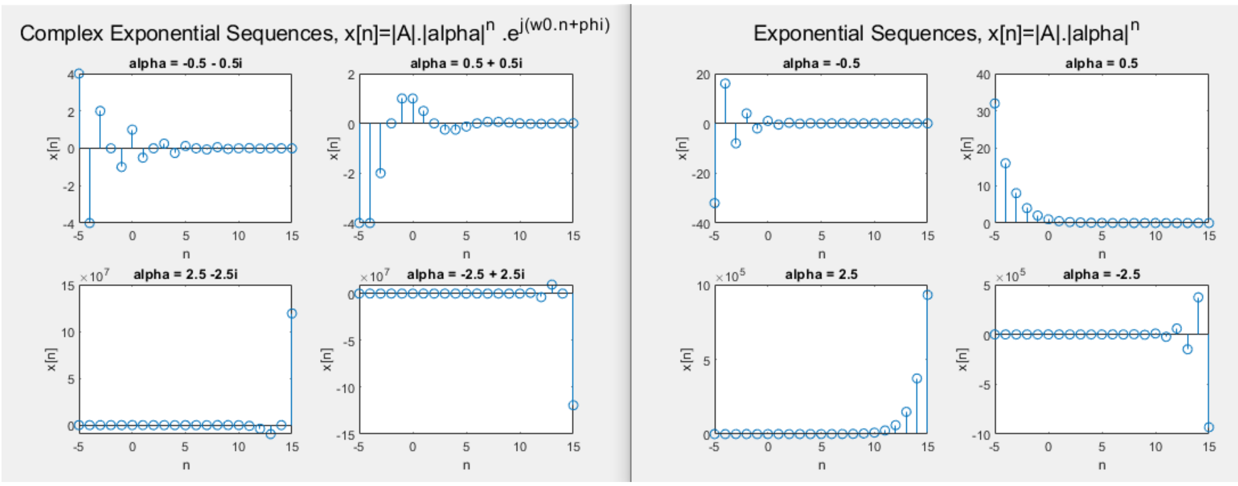
Exponential sequence: $x[n] = A\,\boldsymbol{\alpha}^n$

where $\boldsymbol{\alpha}$ is complex, $x[n] = |A|e^{j\phi}\,|\boldsymbol{\alpha}|e^{j\omega_0 n} = |A||\boldsymbol{\alpha}|^n\,e^{j(\omega_0 n+\phi)}$

$= |A||\boldsymbol{\alpha}|^n(\cos(\omega_0 n+\phi)+j\sin(\omega_0 n+\phi))$

Complex and sinusoidal: $-\boldsymbol{\pi} < \omega_0 < \boldsymbol{\pi}$ or $0 < \omega_0 < 2\boldsymbol{\pi}$.

Exponential sequences for given $\boldsymbol{\alpha}$ (complex $\boldsymbol{\alpha}$ left, real $\boldsymbol{\alpha}$ right):



Periodicity: $x[n] = x[n+N]$, for all n. (definition). Period = N.

Sinusoid: $x[n] = A\cos(\omega_0 n+\phi) = A\cos(\omega_0 n+\omega_0 N+\phi)$

Test: $\omega_0 N = 2\boldsymbol{\pi}k$, (k is integer)

Exponential: $x[n] = e^{j\omega_0(n+N)} = e^{j\omega_0 n}$,

Test: $\omega_0 N = 2\boldsymbol{\pi}k$, (k is integer)

2.2 Discrete-Time Systems

System: Applied transformation $y[n] = T\{x[n]\}$

Memoryless Systems:

Output $y[n_x]$ is only dependent on input $x[n_x]$ where the same index n_x is used for both (no time delay or advance).

Linear Systems: Adherence to superposition. The additive property and scaling property.

Additive property: Where $y_1[n] = T\{x_1[n]\}$ and $y_2[n] = T\{x_2[n]\}$,
$y_2[n] + y_1[n] = T\{x_1[n] + x_2[n]\}$.

Scaling property: $T\{a.x[n]\} = a.y[n]$

Time-Invariant Systems:

Time shift of input causes equal time shift of output. $T\{x[n-M]\} = y[n-M]$

Causality:

The system is causal if output $y[n]$ is only dependent on $x[n+M]$ where $M<=0$.

Stability:

Input x[n] and Output y[n] of system reach a maximum of a number less than infinity. Must hold for all values of n.

2.3 LTI Systems

Two Properties: Linear & Time-Invariant follows:

"Response" $h_k[n]$ describes how system behaves to impulse $\boldsymbol{\delta}$[n-k] occurring at n = k.

$y[n] = T\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$.

➔ Convolution Sum: y[n] = x[n]*h[n].

Performing Discrete-Time convolution sum:

1. Identify bounds of x[k] (where x[k] is non-zero) as N_1 and N_2.
2. Determine expression for x[k]h[n-k].
3. Solve for $y[n] = \sum_{k=N_1}^{N_2} x[k]\, h[n-k]$.

   General solution for exponential (else use tables): $\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N1}-\alpha^{N2+1}}{1-\alpha}$.

Graphical solution: superposition of responses $h_k[n]$ for corresponding input x[n].

2.4 Properties of Linear Time-Invariant Systems

As LTI systems are described by convolution…

LTI is commutative: x[n]*h[n] = h[n]*x[n].

… is additive: $x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n] + x[n]*h_2[n]$.

… is associative: $(x[n]*h_1[n])*h_2[n] = x[n]*(h_1[n]*h_2[n])$

LTI is stable if the sum of impulse responses $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$.

… is causal if h[n] = 0 for n<0 (causality definition).

Finite-duration Impulse response (FIR) systems:

Impulse response h[n] has limited non-zero samples. Simple to determine stability (above).

Infinite-duration impulse response (IIR) systems:

Example: $B_h=\sum_{n=0}^{\infty}|a|^n$. If a<1, B_h is stable and (using geom. series) $= \frac{1}{1-|a|} < \infty$.

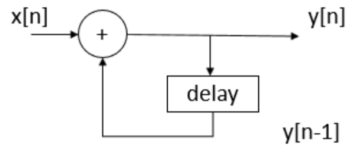
Delay on impulse response: h[n] = sequence*delay = ($\boldsymbol{\delta}$[n+1]- $\boldsymbol{\delta}$[n])* $\boldsymbol{\delta}$[n-1] = $\boldsymbol{\delta}$[n] - $\boldsymbol{\delta}$[n-1].

2.5 Linear Constant-Coefficient Difference Equations

System modifies another version or output of itself to produce new output.

$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} x[n-m]$,

For example: Recursive difference equation block diagram:



As with diff eqns, use homogeneous solution and particular solution. $y[n] = y_p[n] + y_h[n]$.
Where $y_h[n] = \sum_{m=1}^{N} A_m z_m^n$, and A_m chosen for auxiliary y[n] conditions.

2.6 Frequency-Domain Representation of Discrete-Time Signals & Systems

For complex exponentials:

Where $x[n] = e^{j\omega n}$,
and $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$,
$y[n] = H(e^{j\omega})\, e^{j\omega n}$.

$e^{j\omega n}$ is the Eigenfunction of the system.
$H(e^{j\omega})$ is the eigenvalue and frequency response. Always periodic (discrete).

$H(e^{j(\omega+2\pi r)}) = H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j(\omega+2\pi)n}$.

- ➔ Need only specify $H(e^{j\omega})$ over one period $-\pi < \omega < \pi$.
  Low frequencies: closer to zero, high fr at $\pm\pi$.
- ➔ Common to plot Magnitude, Phase for $H(e^{j\omega})$.

Suddenly Applied Complex Exponential Inputs:

Consider inputs of form $x[n] = e^{j\omega n} u[n]$. (zero for n<0)

- ➔ $y[n] = H(e^{j\omega})\, e^{j\omega n} + \left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$.
- ➔ Steady-state response $y_{ss}[n] = H(e^{j\omega})\, e^{j\omega n}$.
- ➔ Transient response $y_t[n] = -\left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$. Def: difference of output from eigenfunction result.

Stability condition also ensures that frequency response function $|H(e^{j\omega})|$ exists:

$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$.

2.7 Representation of Sequences by Fourier Transforms

Series representation by Fourier integral:

$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$, (inverse Fourier transform)
where $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$.
Rectangular: $X(e^{j\omega}) = X_R(e^{j\omega}) + j\, X_I(e^{j\omega})$, Polar or mag,phase: $X(e^{j\omega}) = |X(e^{j\omega})|\, e^{j\angle X(e^{j\omega})}$.

Impulse Response via Fourier Transform integral:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega.$$

Condition: Convergence, i.e. $|X(e^{j\omega})| <= \sum_{k=-\infty}^{\infty} |x[n]| < \infty$,

➔ $X(e^{j\omega})$ exists.

2.8 Symmetry Properties of the Fourier Transform

Conjugate-symmetric sequence: $x_e[n] = x_e^*[-n]$.

Conjugate-antisymmetric sequence: $x_o[n] = -x_o^*[-n]$.

Sequence is sum of above parts: $x[n] = x_o[n] + x_e[n]$.

FT follows: $X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$.

Applicable for general complex sequences:

Sequence x[n]	FT $X(e^{j\omega})$
$x^*[n]$	$X^*(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$Re\{ x[n]\}$	$X_e(e^{j\omega})$
$j.Im\{ x[n]\}$	$X_o(e^{j\omega})$
$x_e[n]$	$X_R(e^{j\omega}) = Re\{ X(e^{j\omega})\}$
$x_o[n]$	$X_{Im}(e^{j\omega}) = j.Im\{ X(e^{j\omega})\}$

2.9 Fourier Transform Theorems

Fourier Transform notation:

$X(e^{j\omega}) = F\{x[n]\}$
$x[n] = F^{-1}\{ X(e^{j\omega})\}$
$x[n] \leftarrow F \rightarrow X(e^{j\omega})$.

Linearity of Fourier Transform:

$ax[n] + by[n] = a X(e^{j\omega}) + b Y(e^{j\omega})$.

Time Shifting, Frequency Shifting Theorem:

$x[n-m] = e^{j\omega m} X(e^{j\omega})$.
$e^{j\omega 0n}x[n] \leftarrow F \rightarrow X(e^{j(\omega - \omega 0)})$.

Time Reversal Theorem:

$x[-n] \leftarrow F \rightarrow X(e^{-j\omega})$.

If x[n] is real, then $x[-n] \leftarrow F \rightarrow X^*(e^{j\omega})$.

Differentiation in Frequency Theorem:

$$n\, x[n] \overset{F}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}.$$

Parseval's Theorem:

Energy Density Spectrum: $|X(e^{j\omega})|^2$

$$E = \sum_{n=-\infty}^{\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega.$$

Convolution Theorem:

Where $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = x[n]*h[n]$, (def. convolution)

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega}).$$

Modulation or Windowing Theorem:

Where $y[n] = x[n]\, w[n]$ (multiplication),

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\vartheta}\right)W\left(e^{j(\omega-\vartheta)}\right)d\vartheta. \quad \text{(periodic convolution)}$$

See table 2.3 in book for Fourier Transform Pairs (page 62).

2.10 Discrete-Time Random Signals

Modeling sequence as random signal. Signal is member of ensemble of D-Time signals. Individual sample x[n] is an outcome of random variable x_n. Collection of random variables is a *random process*. Random signals are not summable and have no direct Fourier transform, but can be summarized in terms of an autocorrelation and/or autocovariance sequence and FT may be used for these. Fourier transform of the autocorrelation sequence: useful for frequency distribution of power in signal, effect of the system on the autocorrelation sequence.

Input signal characterized by mean m_x and autocorrelation function $\phi_{xx}[m]$. Or, there is 1st or 2nd order probability distributions.

Means of input and output:

$m_{x_n} = E\{x_n\}$, $m_{y_n} = E\{y_n\}$, which follows:

$m_x[n] = E\{x[n]\}$ $m_y[n] = E\{y[n]\}$.

If x[n] is stationary, $m_x[n] = m_x$.

Mean of output process: $m_y[n] = E\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k]\, E\{x[n-k]\} = m_x \sum_{k=-\infty}^{\infty} h[k]$

Output being non-stationary, Autocorrelation of output process:

$$\begin{aligned}\phi_{xx}[n,n+m] &= E\{y[n]y[n+m]\}\\ &= \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r] E\{x[n-k]x[n+m-r]\}.\end{aligned}$$

Deterministic Autocorrelation sequence, autocorrelation of h[n] in aperiodic, finite energy sequence: $c_{hh}[\boldsymbol{\ell}] = \sum_{k=-\infty}^{\infty} h[k]h[\ell + k]$.

Fourier Transform for random input: $\mathbf{\Phi}_{yy}(e^{j\omega}) = C_{hh}(e^{j\omega})\ \mathbf{\Phi}_{xx}(e^{j\omega}) = |H(e^{j\omega})|^2\ \mathbf{\Phi}_{xx}(e^{j\omega})$.

Power Density Spectrum:

$$E\{y^2[n]\} = \phi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H\left(e^{j\omega}\right)\right|^2\ \Phi xx(e^{j\omega}) = \text{total power in output.}$$